On the trivectors of a 6-dimensional symplectic vector space

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Abstract

Let V be a 6-dimensional vector space over a field $\mathbb{F}$, let f be a nondegenerate alternating bilinear form on V and let $Sp(V,f) \cong Sp_6(\mathbb{F})$ denote the symplectic group associated with (V,f). The group $GL(V)$ has a natural action on the third exterior power $\bigwedge^3 V$ of V and this action defines 5 families of nonzero trivectors of V (four of whose are orbits for any choice of $\mathbb{F}$). In this paper, we divide three of these 5 families into orbits for the action of $Sp(V,f) \subseteq GL(V)$ on $\bigwedge^3 V$.

Keywords: symplectic group, exterior power, hyperbolic basis
MSC2000: 15A75, 15A63

1 Introduction

1.1 k-vectors

Let $n \in \mathbb{N} \setminus \{0\}$ and $k \in \{1,\ldots,n\}$. Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $\bigwedge^k V$ denote the k-th exterior power of V. The elements of $\bigwedge^k V$ are called the *k-vectors* of V. A k-vector of V is called *decomposable* if it is of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ for some vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$.

For every $\theta \in GL(V)$, there exists a unique $\bigwedge^k(\theta) \in GL(\bigwedge^k V)$ such that $\bigwedge^k(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \bar{v}_k) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \cdots \wedge \theta(\bar{v}_k)$, $\forall \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \in V$. Two k-vectors α_1 and α_2 of V are called *equivalent* if there exists a $\theta \in GL(V)$ for which $\bigwedge^k(\theta)(\alpha_1) = \alpha_2$.

The smallest value of k for which the classification of the (equivalence classes of) k-vectors is nontrivial is the case $k = 3$. Classification results regarding trivectors were obtained by several authors.

- Suppose V is an n-dimensional complex vector space. A classification of the trivectors of V was obtained in Reichel [14] for the case $n = 6$, in Schouten [17] for the case $n = 7$, in Gurevich [10] for the case $n = 8$ and in Vinberg & Èlašvili [18] for the case $n = 9$. A summary of the results obtained for the cases $n \in \{6,7,8\}$ can be found in Gurevich [11, §35].

- Suppose V is an n-dimensional real vector space. A classification of the trivectors of V was obtained in Gurevich [8, 9] and Capdevielle [1] for the case $n = 6$, in Westwick [19] for the case $n = 7$ and in Djoković [7] for the case $n = 8$.

- Suppose V is a vector space of dimension $n \in \{6, 7\}$ over a perfect field of cohomological dimension at most 1. A classification of the trivectors of V was obtained in Cohen & Helminck [2].

- Suppose V is a vector space over an arbitrary field $\mathbb{F}$. A classification of the trivectors of V was obtained in Revoy [15] for the case $n = 6$ and in Revoy [16] for the case $n = 7$.

The case of interest in the present paper is the case of the classification of the trivectors of a 6-dimensional vector space. We will state this classification result in the following subsection.

1.2 The classification of the trivectors of a 6-dimensional vector space

Suppose V is a 6-dimensional vector space over a field $\mathbb{F}$. Let $B^* = (\bar{e}_1^*, \bar{e}_2^*, \ldots, \bar{e}_6^*)$ be a given ordered basis of V and let $\overline{\mathbb{F}}$ denote a fixed algebraic closure of $\mathbb{F}$. For every quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, we will now define a certain trivector $\alpha^*_{\mathbb{F}_1}$ of V. The field $\mathbb{F}_1$ can be regarded as the splitting field of some irreducible quadratic polynomial $q(X) = X^2 - aX - b \in \mathbb{F}[X]$. Since $b = -q(0) \neq 0 \neq q(1) = 1 - a - b$, the values $\mu_1 := a + b - 1$ and $\mu_2 := \frac{1-a-b}{b}$ are nonzero. The field $\mathbb{F}_1$ is also the splitting field of the quadratic polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$. Now, define

$$\alpha^*_{\mathbb{F}_1} \quad := \quad \mu_1 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \mu_2 \cdot \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^* + (\bar{e}_1^* + \bar{e}_4^*) \wedge (\bar{e}_2^* + \bar{e}_5^*) \wedge (\bar{e}_3^* + \bar{e}_6^*).$$

Proposition 1.1 (Revoy [15]) • *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are two distinct quadratic extensions of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, then $\alpha^*_{\mathbb{F}_1}$ and $\alpha^*_{\mathbb{F}_2}$ are not equivalent.*
• *Every nonzero trivector of V is equivalent with precisely one of the following vectors:*
(A) $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;
(B) $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_4^* \wedge \bar{e}_5^*$;
(C) $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_4^* \wedge \bar{e}_5^* \wedge \bar{e}_6^*$;
(D) $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_4^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{e}_5^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{e}_6^*$;
(E) $\alpha^*_{\mathbb{F}_1}$ *for some quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$.*

Let $X \in \{A, B, C, D, E\}$. A trivector α of V is said to be of *Type (X)* if it is equivalent to the trivector which was described in (X) of Proposition 1.1.

The description of the trivector $\alpha^*_{\mathbb{F}_1}$ given above is taken from De Bruyn [6] and is more symmetric than the descriptions given in [2] and [15], where a distinction has been made between the case where the extension $\mathbb{F}_1/\mathbb{F}$ is separable and the case where this extension is not separable.

1.3 The main results

We continue with the notation introduced in Section 1.2. Suppose f is a nondegenerate alternating bilinear form on V. If $\bar{v}$ is a vector of V, then $\bar{v}^\perp$ denotes the subspace of V consisting of all vectors $\bar{w} \in V$ for which $f(\bar{v}, \bar{w}) = 0$. If U is a set of vectors of V, then $U^\perp$ denotes the subspace $\bigcap_{\bar{u} \in U} \bar{u}^\perp$ of V.

If $B = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_6)$ is an ordered basis of V and $\theta \in GL(V)$, then $\theta(B)$ denotes the ordered basis $(\theta(\bar{b}_1), \theta(\bar{b}_2), \ldots, \theta(\bar{b}_6))$ of V. An ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of V is called a *hyperbolic basis* of (V, f) if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, 2, 3\}$. Here, δ_{ij} denotes the Kronecker delta. Let $Sp(V, f) \cong Sp_6(\mathbb{F})$ denote the symplectic group associated with (V, f), i.e. $Sp(V, f)$ consists of all $\theta \in GL(V)$ for which $f(\theta(\bar{v}_1), \theta(\bar{v}_2)) = f(\bar{v}_1, \bar{v}_2)$, $\forall \bar{v}_1, \bar{v}_2 \in V$. The elements of $Sp(V, f)$ are precisely those elements of $GL(V)$ which map hyperbolic bases of (V, f) to hyperbolic bases of (V, f).

Two trivectors α_1 and α_2 of V are called *Sp(V,f)-equivalent* if there exists a $\theta \in Sp(V, f)$ such that $\bigwedge^3(\theta)(\alpha_1) = \alpha_2$. Clearly, if α_1 and α_2 are $Sp(V, f)$-equivalent, then α_1 and α_2 are also equivalent. This means that each equivalence class of trivectors splits into a number of $Sp(V, f)$-equivalence classes.

The present paper is about the problem of classifying the $Sp(V, f)$-equivalence classes of trivectors of V. We give a complete classification of those $Sp(V, f)$-equivalence classes which contain trivectors of Type (A), (B) or (C). We were not (yet) able to deal with those $Sp(V, f)$-equivalence classes which contain trivectors of Type (D) or (E). This problem seems to be much harder. One of our motivations for classifying $Sp(V, f)$-equivalence classes of trivectors came from the study of hyperplanes of symplectic dual polar spaces.

We will now state the main results.

Theorem 1.2 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Then every trivector of Type (A) of V is $Sp(V, f)$-equivalent with precisely one of the following trivectors:*

$(A1)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;
$(A2)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$.

A trivector of V is said to be of *Type (Ai)*, $i \in \{1, 2\}$, if it is $Sp(V, f)$-equivalent with the trivector described in (Ai) of Theorem 1.2.

Theorem 1.3 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basic of (V, f). Then every trivector of Type (B) of V is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

$(B1)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$;
$(B2)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$;
$(B3)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;
$(B4)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;
$(B5)$ $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$.

A trivector of V is said to be of *Type (Bi)*, $i \in \{1, 2, \ldots, 5\}$, if it is $Sp(V, f)$-equivalent with a trivector described in (Bi) of Theorem 1.3.

Theorem 1.4 (1) *Let $i, j \in \{1, 2, \ldots, 5\}$ with $i \neq j$. Then no trivector of Type (Bi) is $Sp(V, f)$-equivalent with a trivector of Type (Bj).*

(2) *Let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$ and let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basic of (V, f). Then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda' \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ are $Sp(V, f)$-equivalent if and only if $\frac{\lambda}{\lambda'}$ is a square in $\mathbb{F}$.*

(3) *Let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$ and let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basic of (V, f). Then the trivectors $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ and $\lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ are $Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

Theorem 1.5 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Then every trivector of Type (C) of V is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

$(C1)$ *$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*

$(C2)$ *$\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*

$(C3)$ *$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*

$(C4)$ *$\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*

$(C5)$ *$\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*

$(C6)$ *(only if $|\mathbb{F}| > 2$) $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$ and some $\epsilon \in \mathbb{F} \setminus \{0, -1\}$.*

A trivector of V is said to be of *Type (Ci)*, $i \in \{1, 2, \ldots, 6\}$, if it is $Sp(V, f)$-equivalent with a trivector described in (Ci) of Theorem 1.5.

Theorem 1.6 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f).*

(1) *Let $i, j \in \{1, 2, \ldots, 6\}$ with $i \neq j$. Then no trivector of Type (Ci) is $Sp(V, f)$-equivalent with a trivector of Type (Cj).*

(2) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda' \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

(3) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$, then the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

(4) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda' \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

(5) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$, then the trivectors $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

(6) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ and $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda' \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

(7) *If $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$ and $\epsilon, \epsilon' \in \mathbb{F} \setminus \{0, -1\}$, then the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon' \cdot \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent if and only if $\epsilon' = \epsilon$ and $\lambda' \in \{\lambda, -\lambda\}$.*

Notice that if $char(\mathbb{F}) = 2$ and $j \in \{2, 3, \ldots, 6\}$, then the two trivectors mentioned in Theorem 1.6(j) are $Sp(V, f)$-equivalent if and only if $\lambda' = \lambda$. If $char(\mathbb{F}) = 2$, then the two trivectors mentioned in Theorem 1.6(7) are $Sp(V, f)$-equivalent if and only if $(\lambda', \epsilon') = (\lambda, \epsilon)$.

2 Some useful lemmas

In this section V denotes a 6-dimensional vector space over a field $\mathbb{F}$ and f denotes a nondegenerate alternating bilinear form on V. Our aim is to derive some properties of hyperbolic bases of (V, f) and k-vectors of V which will prove useful for the determination of some $Sp(V, f)$-equivalence classes of trivectors of V.

Definition 2.1 Let $*$ be a symbol not contained in V. Let $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$ be a hyperbolic basis of (V, f) and let S_i, $i \in \{1, 2, \ldots, 6\}$, be an element of $V \cup \{*\}$. Then we say that $(S_1, S_2, \ldots, S_6)$ can be *extended to* $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$ if $S_i = \bar{v}_i$ for every $i \in \{1, 2, \ldots, 6\}$ for which $S_i \neq *$.

The following lemma has some similarity with Witt's extension theorem (Jacobson [12, p. 369]; Lang [13, p. 591]) for vector spaces equipped with a quadratic form.

Lemma 2.2 *For every $i \in \{1, 2, \ldots, 6\}$, let $S_i \in V \cup \{*\}$. Let I be the set of all $i \in \{1, 2, \ldots, 6\}$ for which $S_i \in V$. Suppose the following holds for all $i_1, i_2 \in I$:*

$$f(S_{i_1}, S_{i_2}) = \begin{cases} 1 & \textit{if } (i_1, i_2) \in \{(1,2), (3,4), (5,6)\}; \\ -1 & \textit{if } (i_1, i_2) \in \{(2,1), (4,3), (6,5)\}; \\ 0 & \textit{otherwise.} \end{cases}$$

If moreover the $|I|$ vectors S_i, $i \in I$, are linearly independent, then $(S_1, S_2, \ldots, S_6)$ can be extended to a hyperbolic basis of (V, f).

Proof. For every $j \in \{1, 2, \ldots, 6\}$, let j' be the element of $\{1, 2, \ldots, 6\}$ such that $\{j, j'\} \in \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. We will prove the lemma by downwards induction on $|I| \in \{0, 1, \ldots, 6\}$. The lemma trivially holds if $|I| = 6$. So, we will suppose that $|I| < 6$. Let i be the smallest element of $\{1, 2, \ldots, 6\}$ which is not contained in I. We distinguish two cases.

(1) Suppose that $i' \notin I$. We prove that $< S_j \,|\, j \in I >^{\perp} \setminus < S_j \,|\, j \in I > \neq \emptyset$, or equivalently that $< S_j \,|\, j \in I >^{\perp} \cap < S_j \,|\, j \in I >$ is properly contained in $< S_j \,|\, j \in I >^{\perp}$. This follows from the fact that the subspace $< S_j \,|\, j \in I >^{\perp}$ has dimension $6 - |I|$, while the subspace $< S_j \,|\, j \in I >^{\perp} \cap < S_j \,|\, j \in I > = < S_j \,|\, j \in I$ and $j' \notin I >$ has dimension at most $4 - |I|$. (There are precisely $4 - |I|$ elements $j \in \{1, 2, \ldots, 6\} \setminus \{i, i'\}$ for which $j' \notin I$.) Now, let S'_i be a vector of $< S_j \,|\, j \in I >^{\perp} \setminus < S_j \,|\, j \in I >$ and put $S'_j := S_j$ for every $j \in \{1, 2, \ldots, 6\} \setminus \{i\}$. Then the $|I| + 1$ vectors S'_j, $j \in I \cup \{i\}$, are linearly independent. By the induction hypothesis, $(S'_1, S'_2, \ldots, S'_6)$ can be extended to a hyperbolic basis of (V, f). Hence, also $(S_1, S_2, \ldots, S_6)$ can be extended to a hyperbolic basis of (V, f).

(2) Suppose that $i' \in I$. Since $S_{i'} \notin < S_j \,|\, j \in I \setminus \{i'\} >$, there exists a vector S_i' in $< S_j \,|\, j \in I \setminus \{i'\} >^{\perp}$ for which $f(S_i', S_{i'}) = 1$ if $i' - i = 1$ and $f(S_i', S_{i'}) = -1$ if $i - i' = 1$. Put $S_j' := S_j$ for every $j \in \{1, 2, \ldots, 6\} \setminus \{i\}$. Since $S_{i'}$ is orthogonal with every vector S_j, $j \in I$, $S_i' \notin < S_j' \,|\, j \in I >$ and hence the $|I| + 1$ vectors S_j', $j \in I \cup \{i\}$, are linearly independent. By the induction hypothesis, $(S_1', S_2', \ldots, S_6')$ can be extended to a hyperbolic basis of (V, f). Hence, also $(S_1, S_2, \ldots, S_6)$ can be extended to a hyperbolic basis of (V, f). ■

Lemma 2.3 *If $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ and $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ are two hyperbolic bases of V, then $\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 = \bar{e}_1' \wedge \bar{f}_1' \wedge \bar{e}_2' \wedge \bar{f}_2' \wedge \bar{e}_3' \wedge \bar{f}_3'$.*

Proof. We have $\bar{e}_1' \wedge \bar{f}_1' \wedge \bar{e}_2' \wedge \bar{f}_2' \wedge \bar{e}_3' \wedge \bar{f}_3' = \det(\eta) \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3$, where η is the element of $Sp(V, f)$ which maps the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ to the hyperbolic basis $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$. The lemma now follows from the fact that $\det(\eta) = 1$ for every $\eta \in Sp(V, f)$, see e.g. Cohn [3, Corollary 3.6.4] or Jacobson [12, p. 393]. ■

Lemma 2.4 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f), let W_B and $\widetilde{W_B}$ denote the following subspaces of $\bigwedge^3 V$:*

$$\begin{aligned} W_B \;=\; & < \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \\ & \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \\ & \bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \\ & \bar{e}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3), \bar{f}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3), \\ & \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2) >, \end{aligned}$$

$$\begin{aligned} \widetilde{W_B} \;=\; & \bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \\ & \bar{e}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_3 \wedge \bar{f}_3), \bar{f}_2 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_3 \wedge \bar{f}_3), \\ & \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2) > . \end{aligned}$$

Then W_B and $\widetilde{W_B}$ are independent of the hyperbolic basis B of (V, f).

Proof. It is known (see e.g. De Bruyn [4, Section 5]) that W_B is the subspace of $\bigwedge^3 V$ generated by all trivectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$, where $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ are three vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is totally isotropic with respect to f. So, W_B must be independent of the hyperbolic basis B of (V, f). The subspace $\widetilde{W_B}$ consists of all trivectors α of V for which $\alpha \wedge \chi = 0$, $\forall \chi \in W_B$. Hence, also this subspace is independent from the chosen hyperbolic basis B of (V, f). ■

Definition 2.5 Put $W := W_B$ and $\widetilde{W} := \widetilde{W_B}$, where B is an arbitrary hyperbolic basis of (V, f).

One can easily verify that all nonzero vectors of $\widetilde{W}$ are $Sp(V, f)$-equivalent, see e.g. De Bruyn [5, Section 3.5]. In fact, every nonzero vector of $\widetilde{W}$ is a trivector of Type (B3). The following is a list of all trivectors of Type (A), (B) and (C) which belong to W:

• the trivectors of Type (A1);
• the trivectors of Type (B3) if $char(\mathbb{F}) = 2$;
• the trivectors of Type (B4);
• the trivectors of Type (C1).

Lemma 2.6 *Suppose α_1 and α_2 are two $Sp(V, f)$-equivalent trivectors of V. Then $\alpha_1 \in W$ if and only if $\alpha_2 \in W$, and $\alpha_1 \in \widetilde{W}$ if and only if $\alpha_2 \in \widetilde{W}$.*

Proof. Let $\theta \in Sp(V, f)$ for which $\alpha_2 = \bigwedge^3(\theta)(\alpha_1)$ and let B be an arbitrary hyperbolic basis of (V, f). Then also $\theta(B)$ is a hyperbolic basis of (V, f). Now, $\alpha_1 \in W \Leftrightarrow \alpha_1 \in W_B \Leftrightarrow \bigwedge^3(\theta)(\alpha_1) \in W_{\theta(B)} \Leftrightarrow \alpha_2 \in W$ and $\alpha_1 \in \widetilde{W} \Leftrightarrow \alpha_1 \in \widetilde{W_B} \Leftrightarrow \bigwedge^3(\theta)(\alpha_1) \in \widetilde{W_{\theta(B)}} \Leftrightarrow \alpha_2 \in \widetilde{W}$. ■

The following lemma is known, see e.g. De Bruyn [4, Section 4].

Lemma 2.7 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f), let π_B denote the linear map from $\bigwedge^3 V$ to V defined by*

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_1, \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_1,$$

$$\pi_B(\bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_2, \pi_B(\bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_2,$$

$$\pi_B(\bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_3, \pi_B(\bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{f}_3.$$

The π_B is independent of the chosen hyperbolic basis B of (V, f).

Definition 2.8 *Put $\pi := \pi_B$ where B is an arbitrary hyperbolic basis of (V, f).*

Lemma 2.9 *Let U be a 4-dimensional vector space over the field $\mathbb{F}$ and let $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4\}$, $\{\bar{u}'_1, \bar{u}'_2, \bar{u}'_3, \bar{u}'_4\}$ be two bases of U such that $\bar{u}_1 \wedge \bar{u}_2 + \bar{u}_3 \wedge \bar{u}_4 = \bar{u}'_1 \wedge \bar{u}'_2 + \bar{u}'_3 \wedge \bar{u}'_4$. Then $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4 = \bar{u}'_1 \wedge \bar{u}'_2 \wedge \bar{u}'_3 \wedge \bar{u}'_4$ and hence the linear map defined by $\bar{u}_i \mapsto \bar{u}'_i$, $i \in \{1, 2, 3, 4\}$, has determinant 1.*

Proof. Observe first that if $char(\mathbb{F}) \neq 2$, then $2 \cdot \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4 = (\bar{u}_1 \wedge \bar{u}_2 + \bar{u}_3 \wedge \bar{u}_4) \wedge (\bar{u}_1 \wedge \bar{u}_2 + \bar{u}_3 \wedge \bar{u}_4) = (\bar{u}'_1 \wedge \bar{u}'_2 + \bar{u}'_3 \wedge \bar{u}'_4) \wedge (\bar{u}'_1 \wedge \bar{u}'_2 + \bar{u}'_3 \wedge \bar{u}'_4) = 2 \cdot \bar{u}'_1 \wedge \bar{u}'_2 \wedge \bar{u}'_3 \wedge \bar{u}'_4$ and hence $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4 = \bar{u}'_1 \wedge \bar{u}'_2 \wedge \bar{u}'_3 \wedge \bar{u}'_4$. The above argument however only works for fields of characteristic distinct from 2. We now give an argument which works for any field.

For every ordered basis $B = (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ of U, let η_B denote the following map from $\bigwedge^2 V$ to $\bigwedge^4 V$: $\beta_{12} \cdot \bar{b}_1 \wedge \bar{b}_2 + \beta_{13} \cdot \bar{b}_1 \wedge \bar{b}_3 + \beta_{14} \cdot \bar{b}_1 \wedge \bar{b}_4 + \beta_{23} \cdot \bar{b}_2 \wedge \bar{b}_3 + \beta_{24} \cdot \bar{b}_2 \wedge \bar{b}_4 + \beta_{34} \cdot \bar{b}_3 \wedge \bar{b}_4 \mapsto (\beta_{12}\beta_{34} - \beta_{13}\beta_{24} + \beta_{14}\beta_{23}) \cdot \bar{b}_1 \wedge \bar{b}_2 \wedge \bar{b}_3 \wedge \bar{b}_4$. If $B = (\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ and $B' = (\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4)$ are two ordered bases of U, then we call B and B' neighboring ordered bases of U (notation: $B \sim B'$) if at least one of the following holds:

• $(\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4) = (\bar{b}_{\sigma(1)}, \bar{b}_{\sigma(2)}, \bar{b}_{\sigma(3)}, \bar{b}_{\sigma(4)})$ for some permutation σ of $\{1, 2, 3, 4\}$;

- $(\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4) = (\lambda \cdot \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;
- $(\bar{b}'_1, \bar{b}'_2, \bar{b}'_3, \bar{b}'_4) = (\bar{b}_1 + \lambda \cdot \bar{b}_2, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ for some $\lambda \in \mathbb{F}$.

One readily verifies that $\eta_B = \eta_{B'}$ if B and B' are two neighboring ordered bases of V. Now, if B and B' are two arbitrary ordered bases of U, then there exist ordered bases $B_0, B_1, \ldots, B_k$ of U (for some $k \in \mathbb{N}$) such that $B \sim B_0 \sim B_1 \sim \cdots \sim B_k = B'$. This implies that $\eta_B = \eta_{B_0} = \eta_{B_1} = \cdots = \eta_{B_k} = \eta_{B'}$.

Now, if we put $B = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)$ and $B' = (\bar{u}'_1, \bar{u}'_2, \bar{u}'_3, \bar{u}'_4)$, then $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4 = \eta_B(\bar{u}_1 \wedge \bar{u}_2 + \bar{u}_3 \wedge \bar{u}_4) = \eta_B(\bar{u}'_1 \wedge \bar{u}'_2 + \bar{u}'_3 \wedge \bar{u}'_4) = \eta_{B'}(\bar{u}'_1 \wedge \bar{u}'_2 + \bar{u}'_3 \wedge \bar{u}'_4) = \bar{u}'_1 \wedge \bar{u}'_2 \wedge \bar{u}'_3 \wedge \bar{u}'_4 = \det(\theta) \cdot \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4$, where θ is the element of $GL(U)$ defined by $\bar{u}_i \mapsto \bar{u}'_i$, $i \in \{1, 2, 3, 4\}$. Hence, $\det(\theta) = 1$. ■

3 Trivectors of Type (A)

As before, let V be a 6-dimensional vector space over a field $\mathbb{F}$ and let f be a nondegenerate alternating bilinear form on V. In the following proposition, we determine all $Sp(V, f)$-equivalence classes of trivectors of Type (A).

Proposition 3.1 *Let $(\bar{e}^*_1, \bar{f}^*_1, \bar{e}^*_2, \bar{f}^*_2, \bar{e}^*_3, \bar{f}^*_3)$ be a given hyperbolic basis of (V, f). Then every trivector of Type (A) of V is $Sp(V, f)$-equivalent with precisely one of the following two trivectors:*

(A1) $\bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{e}^*_3$;

(A2) $\bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2$.

Proof. Let $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$ be a given trivector of Type (A), where $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ are linearly independent vectors of V. We can distinguish two cases:

(i) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is a totally isotropic 3-space of (V, f);

(ii) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is not a totally isotropic 3-space of (V, f).

Suppose case (i) occurs. Put $\bar{e}_1 := \bar{v}_1$, $\bar{e}_2 := \bar{v}_2$ and $\bar{e}_3 := \bar{v}_3$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$. By Lemma 2.2, $(\bar{e}_1, *, \bar{e}_2, *, \bar{e}_3, *)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f).

Suppose case (ii) occurs. Let $\bar{u} \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 > \subseteq \bar{u}^{\perp}$. The vector $\bar{u}$ is determined up to a nonzero factor of $\mathbb{F}$. Let $\bar{e}_2, \bar{f}_2 \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ such that $f(\bar{e}_2, \bar{f}_2) = 1$. Then $< \bar{u}, \bar{e}_2, \bar{f}_2 > = < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. Now, let $\bar{e}_1 \in < \bar{u} >$ such that $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$. By Lemma 2.2, $(\bar{e}_1, *, \bar{e}_2, \bar{f}_2, *, *)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f). ■

Definition 3.2 A trivector of V is said to be of *Type* (Ai), $i \in \{1, 2\}$, if it is $Sp(V, f)$-equivalent with the trivector described in (Ai) of Proposition 3.1.

4 Trivectors of Type (B)

As before, let V be a 6-dimensional vector space over a field $\mathbb{F}$ and let f be a nondegenerate alternating bilinear form on V. In this section, we determine all $Sp(V,f)$-equivalence classes of trivectors of Type (B).

Lemma 4.1 *If $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ are five linearly independent vectors of V, then there exist four vectors $\bar{v}_2'$, $\bar{v}_3'$, $\bar{v}_4'$ and $\bar{v}_5'$ of V such that*

- *$\bar{v}_1$, $\bar{v}_2'$, $\bar{v}_3'$, $\bar{v}_4'$ and $\bar{v}_5'$ are linearly independent;*
- *$\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1 \wedge \bar{v}_4' \wedge \bar{v}_5'$;*
- *$< \bar{v}_1, \bar{v}_2', \bar{v}_3' >$ is contained in $\bar{v}_1^\perp$.*

Proof. The lemma is certainly valid if at least one of the subspaces $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >, < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is contained in $\bar{v}_1^\perp$. So, suppose that neither $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ nor $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is contained in $\bar{v}_1^\perp$. Let $\bar{w}_2$ and $\bar{w}_4$ be nonzero vectors of $\bar{v}_1^\perp$ contained in respectively $< \bar{v}_2, \bar{v}_3 >$ and $< \bar{v}_4, \bar{v}_5 >$. Then there exist vectors $\bar{w}_3 \in < \bar{v}_2, \bar{v}_3 >$ and $\bar{w}_5 \in < \bar{v}_4, \bar{v}_5 >$ such that $\bar{w}_2 \wedge \bar{w}_3 = \bar{v}_2 \wedge \bar{v}_3$ and $\bar{w}_4 \wedge \bar{w}_5 = \bar{v}_4 \wedge \bar{v}_5$. Since neither $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ nor $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is contained in $\bar{v}_1^\perp$, $f(\bar{v}_1, \bar{w}_3) \neq 0 \neq f(\bar{v}_1, \bar{w}_5)$. So, there exists a $\lambda \in \mathbb{F} \setminus \{0\}$ such that $f(\bar{v}_1, \bar{w}_3 - \lambda \cdot \bar{w}_5) = 0$. The lemma now follows from the fact that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{v}_1 \wedge \bar{w}_4 \wedge \bar{w}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge (\bar{w}_3 - \lambda \cdot \bar{w}_5) + \bar{v}_1 \wedge (\bar{w}_4 + \lambda \cdot \bar{w}_2) \wedge \bar{w}_5$ and $< \bar{v}_1, \bar{w}_2, \bar{w}_3 - \lambda \cdot \bar{w}_5 > \subseteq \bar{v}_1^\perp$. ■

Lemma 4.2 *Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is contained in $\bar{v}_1^\perp$, $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not contained in $\bar{v}_1^\perp$ and $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 > = < \bar{v}_1 >$. Then there exist four vectors $\bar{v}_2'$, $\bar{v}_3'$, $\bar{v}_4'$ and $\bar{v}_5'$ of V such that:*

- *$\bar{v}_1$, $\bar{v}_2'$, $\bar{v}_3'$, $\bar{v}_4'$ and $\bar{v}_5'$ are linearly independent;*
- *$\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1 \wedge \bar{v}_4' \wedge \bar{v}_5'$;*
- *$< \bar{v}_1, \bar{v}_2', \bar{v}_3' >$ is a totally isotropic subspace;*
- *$< \bar{v}_1, \bar{v}_4', \bar{v}_5' >$ is not totally isotropic and is contained in $(\bar{v}_4')^\perp$;*
- *$< \bar{v}_2', \bar{v}_3' > \subseteq (\bar{v}_5')^\perp$;*
- *$\bar{v}_2' \in (\bar{v}_4')^\perp$.*

Proof. (a) We first prove the lemma in the special case that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is a totally isotropic subspace. Let $\bar{v}_4'$ denote a vector of $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $< \bar{v}_1, \bar{v}_4, \bar{v}_5 > \subseteq (\bar{v}_4')^\perp$. Notice that $\bar{v}_4'$ is determined up to a nonzero factor of $\mathbb{F}$. Let $\bar{v}_5'$ be a vector of $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $\bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{v}_4' \wedge \bar{v}_5'$. Then $\bar{v}_1 \not\in (\bar{v}_5')^\perp$. So, $(\bar{v}_5')^\perp \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is a 2-space of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ which does not contain $\bar{v}_1$. This 2-space contains a nonzero vector $\bar{v}_2'$ orthogonal with $\bar{v}_4'$. Let $\bar{v}_3'$ be another vector of $(\bar{v}_5')^\perp \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}_1 \wedge \bar{v}_2' \wedge \bar{v}_3'$. Then the vectors $\bar{v}_2'$, $\bar{v}_3'$, $\bar{v}_4'$ and $\bar{v}_5'$ satisfy the conditions of the lemma.

(b) We prove the lemma in the case that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is not a totally isotropic subspace. Let $\bar{w}_4$ be a vector of $< \bar{v}_1, \bar{v}_4, \bar{v}_5 > \setminus < \bar{v}_1 >$ which is orthogonal with $\bar{v}_1$. Since $\bar{w}_4 \not\in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp$, $\bar{w}_4^\perp \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is a 2-space of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ containing $< \bar{v}_1 >$. So, there exists a $\bar{w}_2 \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 > \setminus < \bar{v}_1 >$ not orthogonal with $\bar{w}_4$. Let

$\bar{w}_3 \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and $\bar{w}_5 \in < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3$ and $\bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{w}_4 \wedge \bar{w}_5$. Since $\bar{w}_3 \notin \bar{w}_2^{\perp}$ and $\bar{w}_4 \notin \bar{w}_2^{\perp}$, there exists a $\lambda \in \mathbb{F} \setminus \{0\}$ such that $\bar{w}_3 - \lambda \cdot \bar{w}_4 \in \bar{w}_2^{\perp}$. Now, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{v}_1 \wedge \bar{w}_4 \wedge \bar{w}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge (\bar{w}_3 - \lambda \cdot \bar{w}_4) + \bar{v}_1 \wedge \bar{w}_4 \wedge (\bar{w}_5 - \lambda \cdot \bar{w}_2)$. Now, $< \bar{v}_1, \bar{w}_2, \bar{w}_3 - \lambda \cdot \bar{w}_4 >$ is totally isotropic, $< \bar{v}_1, \bar{w}_2, \bar{w}_3 - \lambda \cdot \bar{w}_4 >^{\perp} = < \bar{v}_1, \bar{w}_2, \bar{w}_3 - \lambda \cdot \bar{w}_4 >$ intersects $< \bar{v}_1, \bar{w}_4, \bar{w}_5 - \lambda \cdot \bar{w}_2 >$ in $< \bar{v}_1 >$ and $< \bar{v}_1, \bar{w}_4, \bar{w}_5 - \lambda \cdot \bar{w}_2 >$ is not contained in $\bar{v}_1^{\perp}$ (since $\bar{w}_2 \in \bar{v}_1^{\perp}$ and $\bar{w}_5 \notin \bar{v}_1^{\perp}$). If we now apply part (a) of the proof, then we see that the lemma is also valid in this case. ■

Lemma 4.3 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). If $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ are five linearly independent vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is contained in $\bar{v}_1^{\perp}$, $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not contained in $\bar{v}_1^{\perp}$ and $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 > = < \bar{v}_1 >$, then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ is Sp(V, f)-equivalent with $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$.*

Proof. By Lemma 4.2, we may suppose that:

- $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is totally isotropic;
- $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not totally isotropic and is contained in $\bar{v}_4^{\perp}$;
- $< \bar{v}_2, \bar{v}_3 > \subseteq \bar{v}_5^{\perp}$;
- $\bar{v}_2 \in \bar{v}_4^{\perp}$.

Notice that $\bar{v}_1 \notin \bar{v}_5^{\perp}$. Put $\bar{e}_1 := \bar{v}_1$, let $\bar{f}_1 \in < \bar{v}_5 >$ such that $f(\bar{e}_1, \bar{f}_1) = 1$ and $\bar{f}_3 \in < \bar{v}_4 >$ such that $\bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_3$. Since $< \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$ is not totally isotropic, $\bar{v}_3 \notin \bar{v}_4^{\perp}$. We take $\bar{e}_3 \in < \bar{v}_3 >$ such that $f(\bar{e}_3, \bar{f}_3) = 1$ and $\bar{e}_2 \in < \bar{v}_2 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$. So, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_3$. Since $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{e}_3) = f(\bar{e}_1, \bar{f}_3) = f(\bar{e}_2, \bar{e}_3) = f(\bar{e}_2, \bar{f}_1) = f(\bar{e}_2, \bar{f}_3) = f(\bar{e}_3, \bar{f}_1) = f(\bar{f}_1, \bar{f}_3) = 0$, $f(\bar{e}_1, \bar{f}_1) = f(\bar{e}_3, \bar{f}_3) = 1$ and $\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{e}_3, \bar{f}_3$ are linearly independent, $(\bar{e}_1, \bar{f}_1, \bar{e}_2, *, \bar{e}_3, \bar{f}_3)$ can be extended to a hyperbolic basis of (V, f) by Lemma 2.2. ■

Lemma 4.4 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). If $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ are five linearly independent vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is contained in $\bar{v}_1^{\perp}$, $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not contained in $\bar{v}_1^{\perp}$ and $\dim(< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 >) = 2$, then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ is Sp(V, f)-equivalent with $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$.*

Proof. Since $\dim(< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 >) = 2$, $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^{\perp} \neq < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and hence $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is not totally isotropic. Since $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not contained in $\bar{v}_1^{\perp}$, there exists a $\bar{w} \in < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $f(\bar{v}_1, \bar{w}) \neq 0$. Let $\bar{e}_2, \bar{f}_2$ be two vectors in $\bar{w}^{\perp} \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ such that $f(\bar{e}_2, \bar{f}_2) = 1$, let $\bar{e}_1 \in < \bar{v}_1 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$, let $\bar{f}_1 \in < \bar{w} >$ such that $f(\bar{e}_1, \bar{f}_1) = 1$, and let $\bar{e}_3$ be a vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 > \cap \bar{f}_1^{\perp}$ such that $\bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3$. Since $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{f}_2) = f(\bar{e}_1, \bar{e}_3) = f(\bar{f}_1, \bar{e}_2) = f(\bar{f}_1, \bar{f}_2) = f(\bar{f}_1, \bar{e}_3) = f(\bar{e}_2, \bar{e}_3) = f(\bar{f}_2, \bar{e}_3) = 0$, $f(\bar{e}_1, \bar{f}_1) = f(\bar{e}_2, \bar{f}_2) = 1$ and $\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3$ are linearly independent, $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, *)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) by Lemma 2.2. ■

Lemma 4.5 *Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $\bar{v}_1^\perp =< \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$ and $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp \neq < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$. Then there exist four vectors $\bar{v}'_2$, $\bar{v}'_3$, $\bar{v}'_4$ and $\bar{v}'_5$ such that:*

- $< \bar{v}_1, \bar{v}'_2, \bar{v}'_3, \bar{v}'_4, \bar{v}'_5 >= \bar{v}_1^\perp$;
- $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}_1 \wedge \bar{v}'_4 \wedge \bar{v}'_5$;
- $< \bar{v}_1, \bar{v}'_2, \bar{v}'_3 >$ *is totally isotropic.*

Proof. The lemma trivially holds if $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is totally isotropic. So, we may suppose that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is not totally isotropic. Since $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp \neq < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$, there exists a vector $\bar{w}_2 \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and a vector $\bar{w}_4 \in < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $\bar{w}_2 \notin \bar{w}_4^\perp$. Let $\bar{w}_3 \in < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and $\bar{w}_5 \in < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3$ and $\bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{w}_4 \wedge \bar{w}_5$. Since $f(\bar{w}_2, \bar{w}_3) \neq 0 \neq f(\bar{w}_2, \bar{w}_4)$, there exists a $\lambda \in \mathbb{F} \setminus \{0\}$ such that $\bar{w}_3 - \lambda \cdot \bar{w}_4 \in \bar{w}_2^\perp$. Now, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{v}_1 \wedge \bar{w}_4 \wedge \bar{w}_5 = \bar{v}_1 \wedge \bar{w}_2 \wedge (\bar{w}_3 - \lambda \cdot \bar{w}_4) + \bar{v}_1 \wedge \bar{w}_4 \wedge (\bar{w}_5 - \lambda \cdot \bar{w}_2)$. If we put $\bar{v}'_2 := \bar{w}_2$, $\bar{v}'_3 := \bar{w}_3 - \lambda \cdot \bar{w}_4$, $\bar{v}'_4 := \bar{w}_4$ and $\bar{v}'_5 := \bar{w}_5 - \lambda \cdot \bar{w}_2$, then we see that the lemma holds. ∎

Lemma 4.6 *Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $\bar{v}_1^\perp =< \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$, $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp =< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ and $f(\bar{v}_2, \bar{v}_3) \neq f(\bar{v}_4, \bar{v}_5)$. Then there exist four vectors $\bar{v}'_2$, $\bar{v}'_3$, $\bar{v}'_4$ and $\bar{v}'_5$ such that*

- $< \bar{v}_1, \bar{v}'_2, \bar{v}'_3, \bar{v}'_4, \bar{v}'_5 >= \bar{v}_1^\perp$;
- $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}_1 \wedge \bar{v}'_4 \wedge \bar{v}'_5$;
- $< \bar{v}_1, \bar{v}'_2, \bar{v}'_3 >$ *is totally isotropic.*

Proof. Let $\lambda \in \mathbb{F} \setminus \{0\}$. We have $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1 \wedge (\bar{v}_2 - \lambda \cdot \bar{v}_4) \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge (\bar{v}_5 + \lambda \cdot \bar{v}_3)$. Since $f(\bar{v}_2 - \lambda \cdot \bar{v}_4, \bar{v}_5 + \lambda \cdot \bar{v}_3) = f(\bar{v}_2, \bar{v}_5) - \lambda \cdot f(\bar{v}_4, \bar{v}_5) + \lambda \cdot f(\bar{v}_2, \bar{v}_3) - \lambda^2 \cdot f(\bar{v}_4, \bar{v}_3) = \lambda \cdot (f(\bar{v}_2, \bar{v}_3) - f(\bar{v}_4, \bar{v}_5)) \neq 0$, $< \bar{v}_1, \bar{v}_2 - \lambda \cdot \bar{v}_4, \bar{v}_3 >^\perp \neq < \bar{v}_1, \bar{v}_4, \bar{v}_5 + \lambda \cdot \bar{v}_3 >$. The lemma now follows from Lemma 4.5. ∎

Lemma 4.7 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $\bar{v}_1^\perp =< \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$, $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp =< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ and $f(\bar{v}_2, \bar{v}_3) = f(\bar{v}_4, \bar{v}_5)$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ is $Sp(V, f)$-equivalent with $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$.*

Proof. Since $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >^\perp =< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$, the subspaces $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ are not totally isotropic. Put $\bar{e}_2 := \bar{v}_2$ and $\bar{e}_3 := \bar{v}_4$. Let $\bar{f}_2 \in < \bar{v}_3 >$ and $\bar{f}_3 \in < \bar{v}_5 >$ such that $f(\bar{e}_2, \bar{f}_2) = f(\bar{e}_3, \bar{f}_3) = 1$. Since $f(\bar{v}_2, \bar{v}_3) = f(\bar{v}_4, \bar{v}_5)$, there exists a $\lambda \in \mathbb{F} \setminus \{0\}$ such that $\bar{v}_3 = \lambda \cdot \bar{f}_2$ and $\bar{v}_5 = \lambda \cdot \bar{f}_3$. If we put $\bar{e}_1 = \lambda \cdot \bar{v}_1$, we have $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3$. Since $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{f}_2) = f(\bar{e}_1, \bar{e}_3) = f(\bar{e}_1, \bar{f}_3) = f(\bar{e}_2, \bar{e}_3) = f(\bar{e}_2, \bar{f}_3) = f(\bar{f}_2, \bar{e}_3) = f(\bar{f}_2, \bar{f}_3) = 0$, $f(\bar{e}_2, \bar{f}_2) = f(\bar{e}_3, \bar{f}_3) = 1$ and $\bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$ are linearly independent, $(\bar{e}_1, *, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) by Lemma 2.2. ∎

Lemma 4.8 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $\bar{v}_1^\perp =< \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$ and $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >, < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ are totally isotropic. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ is $Sp(V, f)$-equivalent with $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.*

Proof. Put $\bar{e}_1 := \bar{v}_1$, $\bar{e}_2 := \bar{v}_2$ and $\bar{e}_3 := \bar{v}_3$. For every $i \in \{2,3\}$, let A_i be the two-space $\bar{e}_i^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$. If $\bar{u} \in (A_2 \cap A_3) \setminus < \bar{v}_1 >$, then $< \bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ would be a totally isotropic 4-space, a contradiction. Hence, $A_2 \cap A_3 =< \bar{v}_1 >$. Let $\bar{f}_2 \in A_3 \setminus < \bar{v}_1 >$ and $\bar{f}_3 \in A_2 \setminus < \bar{v}_1 >$ such that $f(\bar{e}_2, \bar{f}_2) = f(\bar{e}_3, \bar{f}_3) = 1$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. Since $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{e}_3) = f(\bar{e}_1, \bar{f}_2) = f(\bar{e}_1, \bar{f}_3) = f(\bar{e}_2, \bar{e}_3) = f(\bar{e}_2, \bar{f}_3) = f(\bar{e}_3, \bar{f}_2) = f(\bar{f}_2, \bar{f}_3) = 0$, $f(\bar{e}_2, \bar{f}_2) = f(\bar{e}_3, \bar{f}_3) = 1$ and $\bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$ are linearly independent, $(\bar{e}_1, *, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) by Lemma 2.2. ■

Lemma 4.9 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ be five linearly independent vectors of V such that $\bar{v}_1^{\perp} =< \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5 >$, $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is totally isotropic and $< \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is not totally isotropic. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ is $Sp(V, f)$-equivalent with $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.*

Proof. Let $\bar{w}_1 \in < \bar{v}_1 >$, $\bar{e}_2 \in < \bar{v}_4 >$ and $\bar{f}_2 \in \langle \bar{v}_5 \rangle$ such that $\bar{w}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ and $f(\bar{e}_2, \bar{f}_2) = 1$. Put $A := \bar{e}_2^{\perp} \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and $B := \bar{f}_2^{\perp} \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. Then A and B are two 2-spaces containing $< \bar{v}_1 >$. We prove that $A \neq B$.

Suppose to the contrary that $A = B$. Let $\bar{w} \in < \bar{v}_2, \bar{v}_3 >$ such that $\bar{w} \notin A = B$. Then every vector of $\bar{w}^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 >$ is orthogonal with every vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. If $\bar{u} \in (\bar{w}^{\perp} \cap < \bar{v}_1, \bar{v}_4, \bar{v}_5 >) \setminus < \bar{v}_1 >$, then $< \bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ would be a totally isotropic 4-space, a contradiction. Hence, $A \neq B$.

Now, let $\bar{w}_2 \in A$ and $\bar{w}_3 \in B$ such that $f(\bar{w}_2, \bar{f}_2) = 1$ and $f(\bar{w}_3, \bar{e}_2) = -1$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \frac{1}{\lambda} \cdot \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. Put $\bar{e}_3 := \bar{e}_2 - \bar{w}_2$ and $\bar{f}_3 := \bar{w}_3 - \bar{f}_2$. Then $f(\bar{e}_3, \bar{f}_3) = f(\bar{e}_2, \bar{w}_3) - f(\bar{e}_2, \bar{f}_2) - f(\bar{w}_2, \bar{w}_3) + f(\bar{w}_2, \bar{f}_2) = 1 - 1 - 0 + 1 = 1$. Put $\bar{e}_1 := \frac{\bar{w}_1}{\lambda}$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge (\bar{e}_2 - \bar{e}_3) \wedge (\bar{f}_2 + \bar{f}_3)$. Since $f(\bar{e}_1, \bar{e}_2) = f(\bar{e}_1, \bar{e}_3) = f(\bar{e}_1, \bar{f}_2) = f(\bar{e}_1, \bar{f}_3) = f(\bar{e}_2, \bar{e}_3) = f(\bar{e}_2, \bar{f}_3) = f(\bar{e}_3, \bar{f}_2) = f(\bar{f}_2, \bar{f}_3) = 0$, $f(\bar{e}_2, \bar{f}_2) = f(\bar{e}_3, \bar{f}_3) = 1$ and $\bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$ are linearly independent, $(\bar{e}_1, *, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) by Lemma 2.2. ■

The following corollary of Lemmas 4.1, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9 is precisely Theorem 1.3.

Corollary 4.10 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Then every trivector of Type (B) of V is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

$(B1)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$;
$(B2)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$;
$(B3)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;
$(B4)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;
$(B5)$ $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$.

Definition 4.11 A trivector of V is said to be of *Type (Bi)*, $i \in \{1, 2, \ldots, 5\}$, if it is $Sp(V, f)$-equivalent with the trivector described in (Bi) of Corollary 4.10.

Lemma 4.12 *Let $i \in \{1,2,4,5\}$. Then no trivector of Type $(B3)$ is $Sp(V,f)$-equivalent with a trivector of Type (Bi).*

Proof. Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda \in \mathbb{F} \setminus \{0\}$. The trivector $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ belongs to the subspace $\widetilde{W}$, while none of the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$, $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$, $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$, $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ belongs to $\widetilde{W}$. The lemma now follows from Lemma 2.6. ■

Lemma 4.13 *No trivector of Type $(B4)$ is $Sp(V,f)$-equivalent with a trivector of Type $(B5)$.*

Proof. Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$. The trivector $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ belongs to W, while the trivector $\lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ does not belong to W. The lemma now follows from Lemma 2.6. ■

Lemma 4.14 *If $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5\}$ and $\{\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5'\}$ are two sets of five linearly independent vectors of V such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1' \wedge \bar{v}_4' \wedge \bar{v}_5'$, then $<\bar{v}_1> = <\bar{v}_1'>$ and $<\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5> = <\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5'>$.*

Proof. By De Bruyn [6, Section 7.3], the set of all vectors $\bar{x}$ of V for which $(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5) \wedge \bar{x} = \bar{o}$ coincides with the subspace $<\bar{v}_1>$. Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1' \wedge \bar{v}_4' \wedge \bar{v}_5'$, this set is also equal to $<\bar{v}_1'>$. By De Bruyn [6, Section 7.3], the set of all vectors $\bar{x}$ of V for which $(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5) \wedge \bar{x}$ is decomposable coincides with the subspace $<\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5>$. Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1' \wedge \bar{v}_4' \wedge \bar{v}_5'$, this set is also equal to $<\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5'>$. ■

Lemma 4.15 *No trivector of Type $(B1)$ is $Sp(V,f)$-equivalent with a trivector of Type $(B2)$.*

Proof. Let $B^* = (\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V,f). Suppose the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$ are $Sp(V,f)$-equivalent. Then there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V,f) such that $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^* = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3$. By Lemma 4.14, $<\bar{e}_1^*> = <\bar{e}_1>$ and $<\bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*, \bar{f}_1^*, \bar{f}_3^*> = <\bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{f}_1, \bar{e}_3>$. Hence, $<\bar{e}_2^*> = <\bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*, \bar{f}_1^*, \bar{f}_3^*>^\perp = <\bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{f}_1, \bar{e}_3>^\perp = <\bar{e}_3>$. We also have $\bar{f}_3^* = \pi_{B^*}(\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*) = \pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3) = \bar{e}_1 + \bar{e}_3$. Hence, $\bar{f}_3^* \in <\bar{e}_1^*, \bar{e}_2^*>$, a contradiction. ■

Lemma 4.16 *Suppose $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5\}$ and $\{\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5'\}$ are two sets of five linearly independent vectors of V. Put $\alpha = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ and $\alpha' := \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' + \bar{v}_1' \wedge \bar{v}_4' \wedge \bar{v}_5'$. If α and α' are $Sp(V,f)$-equivalent, then $\dim(\bar{v}_1^\perp \cap <\bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5>) = \dim((\bar{v}_1')^\perp \cap <\bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5'>)$.*

Proof. Suppose α and α' are $Sp(V,f)$-equivalent. Then there exists a $\theta \in Sp(V,f)$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5 = \theta(\bar{v}_1') \wedge \theta(\bar{v}_2') \wedge \theta(\bar{v}_3') + \theta(\bar{v}_1') \wedge \theta(\bar{v}_4') \wedge \theta(\bar{v}_5')$. By Lemma

4.14, $<\bar{v}_1>=<\theta(\bar{v}'_1)>$ and $<\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5>=<\theta(\bar{v}'_1),\theta(\bar{v}'_2),\theta(\bar{v}'_3),\theta(\bar{v}'_4),\theta(\bar{v}'_5)>$. Hence, $\dim(\bar{v}_1^{\perp}\cap<\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5>)=\dim(\theta(\bar{v}'_1)^{\perp}\cap<\theta(\bar{v}'_1),\theta(\bar{v}'_2),\theta(\bar{v}'_3),\theta(\bar{v}'_4),\theta(\bar{v}'_5)>)=\dim(\theta((\bar{v}'_1)^{\perp})\cap<\theta(\bar{v}'_1),\theta(\bar{v}'_2),\theta(\bar{v}'_3),\theta(\bar{v}'_4),\theta(\bar{v}'_5)>)=\dim((\bar{v}'_1)^{\perp}\cap<\bar{v}'_1,\bar{v}'_2,\bar{v}'_3,\bar{v}'_4,\bar{v}'_5>)$. This implies that $\dim(\bar{v}_1^{\perp}\cap<\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5>)=\dim((\bar{v}'_1)^{\perp}\cap<\bar{v}'_2,\bar{v}'_3,\bar{v}'_4,\bar{v}'_5>)$. ∎

The following is an immediate corollary of Lemma 4.16.

Corollary 4.17 *Let $i\in\{1,2\}$ and $j\in\{4,5\}$. Then no trivector of Type (Bi) is $Sp(V,f)$-equivalent with a trivector of Type (Bj).*

Lemma 4.18 *Let $(\bar{e}_1^*,\bar{f}_1^*,\bar{e}_2^*,\bar{f}_2^*,\bar{e}_3^*,\bar{f}_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda,\lambda'\in\mathbb{F}\setminus\{0\}$. Then $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ and $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda'\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ are $Sp(V,f)$-equivalent if and only if $\frac{\lambda'}{\lambda}$ is a square in $\mathbb{F}$.*

Proof. Suppose $\frac{\lambda'}{\lambda}=\mu^2$ where $\mu\in\mathbb{F}$. Since $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda'\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*=(\mu\cdot\bar{e}_1^*)\wedge(\frac{\bar{e}_2^*}{\mu})\wedge\bar{e}_3^*+\lambda\cdot(\mu\cdot\bar{e}_1^*)\wedge(\mu\cdot\bar{f}_2^*)\wedge\bar{f}_3^*$ and $(\mu\cdot\bar{e}_1^*,\frac{\bar{f}_1^*}{\mu},\frac{\bar{e}_2^*}{\mu},\mu\cdot\bar{f}_2^*,\bar{e}_3^*,\bar{f}_3^*)$ is a hyperbolic basis of (V,f), the trivectors $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ and $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda'\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ are $Sp(V,f)$-equivalent.

Conversely, suppose that the trivectors $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ and $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda'\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ are $Sp(V,f)$-equivalent. Then there exists a hyperbolic basis $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ of (V,f) such that $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*=\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\lambda'\cdot\bar{e}_1\wedge\bar{f}_2\wedge\bar{f}_3$. By Lemma 4.14, $<\bar{e}_1^*>=<\bar{e}_1>$ and $<\bar{e}_1^*,\bar{e}_2^*,\bar{e}_3^*,\bar{f}_2^*,\bar{f}_3^*>=<\bar{e}_1,\bar{e}_2,\bar{e}_3,\bar{f}_2,\bar{f}_3>$. Let $\mu\in\mathbb{F}\setminus\{0\}$ and $\mu_1,\mu_2,\mu_3,\mu_4\in\mathbb{F}$ such that $\bar{e}_1=\mu\cdot\bar{e}_1^*$ and such that each of the vectors $\bar{e}'_2:=\bar{e}_2+\mu_1\cdot\bar{e}_1$, $\bar{e}'_3:=\bar{e}_3+\mu_2\cdot\bar{e}_1$, $\bar{f}'_2:=\bar{f}_2+\mu_3\cdot\bar{e}_1$, $\bar{f}'_3:=\bar{f}_3+\mu_4\cdot\bar{e}_1$ belongs to $<\bar{e}_2^*,\bar{e}_3^*,\bar{f}_2^*,\bar{f}_3^*>$. Then $(\bar{e}_1^*,\bar{f}_1^*,\bar{e}'_2,\bar{f}'_2,\bar{e}'_3,\bar{f}'_3)$ is a hyperbolic basis of (V,f) and $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{e}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*=\mu\cdot\bar{e}_1^*\wedge\bar{e}'_2\wedge\bar{e}'_3+(\mu\lambda')\cdot\bar{e}_1^*\wedge\bar{f}'_2\wedge\bar{f}'_3$. So, $\bar{e}_1^*\wedge(\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{f}_2^*\wedge\bar{f}_3^*-\mu\cdot\bar{e}'_2\wedge\bar{e}'_3-(\mu\lambda')\cdot\bar{f}'_2\wedge\bar{f}'_3)=0$ and hence $\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{f}_2^*\wedge\bar{f}_3^*=\mu\cdot\bar{e}'_2\wedge\bar{e}'_3+(\mu\lambda')\cdot\bar{f}'_2\wedge\bar{f}'_3$. By Lemma 2.9 applied to the subspace $<\bar{e}_2^*,\bar{f}_2^*,\bar{e}_3^*,\bar{f}_3^*>$, $\lambda\cdot\bar{e}_2^*\wedge\bar{e}_3^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*=(\mu^2\lambda')\cdot\bar{e}'_2\wedge\bar{e}'_3\wedge\bar{f}'_2\wedge\bar{f}'_3$. Hence, $\lambda\cdot\bar{e}_1^*\wedge\bar{f}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*=(\mu^2\lambda')\cdot\bar{e}_1^*\wedge\bar{f}_1^*\wedge\bar{e}'_2\wedge\bar{e}'_3\wedge\bar{f}'_2\wedge\bar{f}'_3$. By Lemma 2.3, $\lambda=\mu^2\lambda'$. So, $\frac{\lambda'}{\lambda}$ is a square in $\mathbb{F}$. ∎

Lemma 4.19 *Let $B^*=(\bar{e}_1^*,\bar{f}_1^*,\bar{e}_2^*,\bar{f}_2^*,\bar{e}_3^*,\bar{f}_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda,\lambda'\in\mathbb{F}\setminus\{0\}$. Then $\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)$ and $\lambda'\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)$ are $Sp(V,f)$-equivalent if and only if $\lambda=\lambda'$.*

Proof. Suppose the trivectors $\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)$ and $\lambda'\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)$ are $Sp(V,f)$-equivalent. Then there exists a hyperbolic basis $B=(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ of (V,f) such that $\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)=\lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2+\bar{e}_1\wedge(\bar{e}_2-\bar{e}_3)\wedge(\bar{f}_2+\bar{f}_3)$. By Lemma 4.14, $<\bar{e}_1^*>=<\bar{e}_1>$ and $<\bar{e}_1^*,\bar{e}_2^*,\bar{f}_2^*,\bar{e}_3^*,\bar{f}_3^*>=<\bar{e}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3>$. We have $\lambda\cdot\bar{e}_1^*=\pi_{B^*}(\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*))=\pi_B(\lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2+\bar{e}_1\wedge(\bar{e}_2-\bar{e}_3)\wedge(\bar{f}_2+\bar{f}_3))=\lambda'\cdot\bar{e}_1$. Let $\mu_1,\mu_2,\mu_3,\mu_4\in\mathbb{F}$ such that each of the vectors $\bar{e}'_2:=\bar{e}_2+\mu_1\cdot\bar{e}_1$, $\bar{e}'_3:=\bar{e}_3+\mu_2\cdot\bar{e}_1$, $\bar{f}'_2:=\bar{f}_2+\mu_3\cdot\bar{e}_1$, $\bar{f}'_3:=\bar{f}_3+\mu_4\cdot\bar{e}_1$ belongs to $<\bar{e}_2^*,\bar{e}_3^*,\bar{f}_2^*,\bar{f}_3^*>$. Then $(\bar{e}_1^*,\bar{f}_1^*,\bar{e}'_2,\bar{f}'_2,\bar{e}'_3,\bar{f}'_3)$ is a hyperbolic basis of (V,f). Now, $\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*+\bar{e}_1^*\wedge(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*)=\lambda\cdot\bar{e}_1^*\wedge\bar{e}'_2\wedge\bar{f}'_2+\frac{\lambda}{\lambda'}\cdot\bar{e}_1^*\wedge(\bar{e}'_2-\bar{e}'_3)\wedge(\bar{f}'_2+\bar{f}'_3)$.

This implies that $\lambda\cdot\bar{e}_2^*\wedge\bar{f}_2^*+(\bar{e}_2^*-\bar{e}_3^*)\wedge(\bar{f}_2^*+\bar{f}_3^*) = \lambda\cdot\bar{e}_2'\wedge\bar{f}_2'+\frac{\lambda}{\lambda'}\cdot(\bar{e}_2'-\bar{e}_3')\wedge(\bar{f}_2'+\bar{f}_3')$. By Lemma 2.9 applied to the subspace $< \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^* >$, we have $\lambda\cdot\bar{e}_2^*\wedge\bar{f}_2^*\wedge\bar{e}_3^*\wedge\bar{f}_3^* = \frac{\lambda^2}{\lambda'}\cdot\bar{e}_2'\wedge\bar{f}_2'\wedge\bar{e}_3'\wedge\bar{f}_3'$ and hence also $\lambda\cdot\bar{e}_1^*\wedge\bar{f}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*\wedge\bar{e}_3^*\wedge\bar{f}_3^* = \frac{\lambda^2}{\lambda'}\cdot\bar{e}_1^*\wedge\bar{f}_1^*\wedge\bar{e}_2'\wedge\bar{f}_2'\wedge\bar{e}_3'\wedge\bar{f}_3'$. By Lemma 2.3, we then have that $\lambda = \frac{\lambda^2}{\lambda'}$, i.e. $\lambda' = \lambda$. ∎

Theorem 1.4 is a consequence of Lemmas 4.12, 4.13, 4.15, 4.18, 4.19 and Corollary 4.17.

5 Trivectors of Type (C)

As before, let V be a 6-dimensional vector space over a field $\mathbb{F}$ and let f be a nondegenerate alternating bilinear form on V. In this section, we determine all $Sp(V,f)$-equivalence classes of trivectors of Type (C).

Lemma 5.1 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V,f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$ be six linearly independent vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$, $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ are totally isotropic. Then $\bar{v}_1\wedge\bar{v}_2\wedge\bar{v}_3+\bar{v}_4\wedge\bar{v}_5\wedge\bar{v}_6$ is $Sp(V,f)$-equivalent with $\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^*+\lambda\cdot\bar{f}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^*$ for some $\lambda\in\mathbb{F}\setminus\{0\}$.*

Proof. Put $\bar{e}_1 := \bar{v}_1$, $\bar{e}_2 := \bar{v}_2$ and $\bar{e}_3 := \bar{v}_3$. For every $i\in\{1,2,3\}$, put $A_i := \bar{e}_i^{\perp}\cap < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. If $A_1\cap A_2\cap A_3$ would contain a nonzero vector $\bar{u}$, then $< \bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ would be a totally isotropic 4-space, a contradiction. So, we necessarily have $\dim(A_1\cap A_2) = \dim(A_1\cap A_3) = \dim(A_2\cap A_3) = 1$. For every $i\in\{1,2,3\}$, let $\bar{f}_i$ be the unique vector in $\bigcap_{j\in\{1,2,3\}\setminus\{i\}} A_j$ satisfying $f(\bar{e}_i,\bar{f}_i) = 1$. Then $\bar{v}_1\wedge\bar{v}_2\wedge\bar{v}_3+\bar{v}_4\wedge\bar{v}_5\wedge\bar{v}_6 = \bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\lambda\cdot\bar{f}_1\wedge\bar{f}_2\wedge\bar{f}_3$ for some $\lambda\in\mathbb{F}\setminus\{0\}$. Obviously, $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ is a hyperbolic basis of (V,f). ∎

Lemma 5.2 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V,f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$ be six linearly independent vectors of V such that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is totally isotropic and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ is not totally isotropic. Then $\bar{v}_1\wedge\bar{v}_2\wedge\bar{v}_3+\bar{v}_4\wedge\bar{v}_5\wedge\bar{v}_6$ is $Sp(V,f)$-equivalent with $\bar{f}_1^*\wedge(\bar{e}_2^*+\bar{e}_3^*)\wedge(\bar{f}_2^*-\bar{f}_3^*)+\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*$ for some $\lambda\in\mathbb{F}\setminus\{0\}$.*

Proof. Without loss of generality, we may suppose that $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >\subseteq \bar{v}_4^{\perp}$.

Suppose $\bar{u}_1$ is a nonzero vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ orthogonal with $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. Let $\bar{u}_2$ and $\bar{u}_3$ be vectors of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ such that $< \bar{u}_1, \bar{u}_2, \bar{u}_3 >=< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. Clearly, $\bar{u}_2^{\perp}\cap\bar{u}_3^{\perp}\cap < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ contains a nonzero vector $\bar{w}$. But then $< \bar{w}, \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ would be a totally isotropic 4-space, a contradiction. Hence, no nonzero vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is orthogonal with $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$.

Put $\bar{e}_2 := \bar{v}_5$ and let $\bar{f}_2 \in< \bar{v}_6 >$ such that $f(\bar{e}_2,\bar{f}_2) = 1$. For every $i\in\{4,5,6\}$, put $A_i := \bar{v}_i^{\perp}\cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. By the previous paragraph, $A_4\cap A_5\cap A_6 = 0$. Hence, $\dim(A_4\cap A_5) = \dim(A_4\cap A_6) = \dim(A_5\cap A_6) = 1$. Let $\bar{w}_2$ be the unique vector of $A_4\cap A_5$ for which $f(\bar{w}_2,\bar{f}_2) = 1$, let $\bar{w}_3$ be the unique vector of $A_4\cap A_6$ for which $f(\bar{e}_2,\bar{w}_3) = 1$, let $\bar{f}_1$ be the unique vector of $A_5\cap A_6$ for which $\bar{f}_1\wedge\bar{w}_2\wedge\bar{w}_3 = \bar{v}_1\wedge\bar{v}_2\wedge\bar{v}_3$ and let $\bar{e}_1$ be the unique vector of $< \bar{v}_4 >$ for which $f(\bar{e}_1,\bar{f}_1) = 1$. Now, put $\bar{e}_3 := \bar{w}_2-\bar{e}_2$ and $\bar{f}_3 := \bar{f}_2-\bar{w}_3$.

Then $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f) and $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 - \bar{f}_3) + \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. ∎

Lemma 5.3 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a fixed hyperbolic basis of (V, f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$ be 6 linearly independent vectors of V such that neither $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ nor $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ is totally isotropic. Suppose moreover that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 > \subseteq \bar{v}_1^\perp$, $< \bar{v}_4, \bar{v}_5, \bar{v}_6 > \subseteq \bar{v}_4^\perp$ and $\bar{v}_4 \not\in \bar{v}_1^\perp$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

- *$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*
- *$\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$;*
- *(only if $|\mathbb{F}| > 2$) $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F} \setminus \{0\}$ and some $\epsilon \in \mathbb{F} \setminus \{0, -1\}$.*

Proof. Let A be the 2-space $\bar{v}_4^\perp \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ and let B be the 2-space $\bar{v}_1^\perp \cap < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. We distinguish 3 cases.

(I) Suppose that $B \subseteq A^\perp$. Let $\bar{e}_2$ and $\bar{f}_2$ be two vectors of A such that $f(\bar{e}_2, \bar{f}_2) = 1$, let $\bar{e}_3$ and $\bar{f}_3$ be two vectors of B such that $f(\bar{e}_3, \bar{f}_3) = 1$, let $\bar{e}_1 \in < \bar{v}_1 >$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ and let $\bar{f}_1 \in < \bar{v}_4 >$ such that $f(\bar{e}_1, \bar{f}_1) = 1$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \lambda \cdot \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. Obviously, $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f).

(II) Suppose $A^\perp \cap B =< \bar{w}_5 >$ for some nonzero vector $\bar{w}_5$ of B. Let $\bar{w}_6 \in B \setminus < \bar{w}_5 >$. Since $A^\perp \cap B =< \bar{w}_5 >$, $\bar{w}_6^\perp \cap A =< \bar{w}_2 >$ for some nonzero vector $\bar{w}_2$. Let $\bar{w}_3 \in A \setminus < \bar{w}_2 >$. Since $\bar{w}_6 \not\in \bar{w}_3^\perp$, there exists a $\bar{w}_3' \in < \bar{w}_3 >$ and a $\bar{w}_6' \in < \bar{w}_6 >$ such that $f(\bar{w}_3', \bar{w}_6') = -1$. Let $\bar{w}_5' \in < \bar{w}_5 >$ and $\bar{w}_2' \in < \bar{w}_2 >$ such that $f(\bar{w}_2', \bar{w}_3') = f(\bar{w}_5', \bar{w}_6') = 1$. There exist unique vectors $\bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$ of V such that $\bar{w}_2' = \bar{e}_2$, $\bar{w}_3' = \bar{f}_2$, $\bar{w}_5' = \bar{e}_3$ and $\bar{w}_6' = \bar{e}_2 + \bar{f}_3$. Now, let $\bar{f}_1 \in < \bar{v}_4 >$ such that $\bar{f}_1 \wedge \bar{w}_5' \wedge \bar{w}_6' = \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ and let $\bar{e}_1 \in < \bar{v}_1 >$ such that $f(\bar{e}_1, \bar{f}_1) = 1$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{f}_1 \wedge \bar{e}_3 \wedge (\bar{e}_2 + \bar{f}_3) + \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ for some $\lambda \in \mathbb{F} \setminus \{0\}$. One can verify that $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f).

(III) Suppose that $A^\perp \cap B = 0$. By the discussion in (II), we then also know that $B^\perp \cap A = 0$ (since $\dim(B^\perp \cap A) = 1$ implies that $\dim(A^\perp \cap B) = 1$). Let V' be the 4-dimensional subspace $< A, B >$ and let f' be the restriction of f to $V' \times V'$. Since $V' =< A, B >= \{\bar{v}_1, \bar{v}_4\}^\perp$ and $\bar{v}_1 \not\in \bar{v}_4^\perp$, f' is a nondegenerate alternating bilinear form on V'. Let $\bar{v}_5'$ and $\bar{v}_6'$ be two vectors of B such that $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}_4 \wedge \bar{v}_5' \wedge \bar{v}_6'$. Let $\bar{v}_2' \in A \cap (\bar{v}_5')^\perp$ and $\bar{v}_3' \in A \cap (\bar{v}_6')^\perp$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}_1 \wedge \bar{v}_2' \wedge \bar{v}_3'$. Put $\bar{e}_2 := \bar{v}_5'$, let $\bar{f}_2 \in < \bar{v}_6' >$ such that $f(\bar{e}_2, \bar{f}_2) = 1$, let $\bar{w}_2 \in < \bar{v}_2' >$ such that $f(\bar{w}_2, \bar{f}_2) = 1$, let $\bar{w}_3 \in < \bar{v}_3' >$ such that $f(\bar{w}_3, \bar{e}_2) = -1$, let $\bar{f}_1 \in < \bar{v}_1 >$ such that $\bar{f}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{v}_1 \wedge \bar{v}_2' \wedge \bar{v}_3' = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$ and let $\bar{e}_1 \in < \bar{v}_4 >$ such that $f(\bar{e}_1, \bar{f}_1) = 1$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{f}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.

Now, put $\epsilon := f(\bar{w}_2, \bar{w}_3) - 1$. Since $f(\bar{w}_2, \bar{w}_3) \neq 0$, $\epsilon \neq -1$. Suppose $\epsilon = 0$. Then $f(\bar{w}_2, \bar{w}_3) = 1$ and one verifies that $f(\bar{w}_2 - \bar{e}_2, \bar{u}) = 0$ for all $\bar{u} \in < A, B >$. This contradicts the fact that f' is nondegenerate. Hence, $|\mathbb{F}| > 2$ and $\epsilon \in \mathbb{F} \setminus \{0, -1\}$. Now, define $\bar{e}_3 := \bar{w}_2 - \bar{e}_2$ and $\bar{f}_3 := \frac{\bar{w}_3 - \bar{f}_2}{\epsilon}$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \epsilon \cdot \bar{f}_3) + \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ and $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f). ∎

Lemma 5.4 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a fixed hyperbolic basis of (V, f). Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, $\bar{v}_5$ and $\bar{v}_6$ be six linearly independent vectors of V such that neither $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ nor $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ is totally isotropic. Suppose moreover that $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >\subseteq \bar{v}_1^\perp$, $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >\subseteq \bar{v}_4^\perp$ and $\bar{v}_4 \in \bar{v}_1^\perp$. Then $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ is $Sp(V, f)$-equivalent with $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ for some $\lambda \in \mathbb{F} \setminus \{0\}$.*

Proof. Since $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >\subseteq \bar{v}_4^\perp$, $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ cannot be contained in $\bar{v}_4^\perp$. So, $\bar{v}_4^\perp\cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is a 2-space A_1 of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. In a similar way, one proves that $\bar{v}_1^\perp\cap < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ is a 2-space B_1 of $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$.

Suppose $\bar{u}$ is a nonzero vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ orthogonal with $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. By the previous paragraph, $\bar{u} \notin \langle \bar{v}_1 \rangle$. Since $\bar{u}$ is orthogonal with $\bar{v}_4$, $\bar{u} \in A_1\setminus < \bar{v}_1 >$. Since $A_1 \subseteq \bar{u}^\perp$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >\subseteq \bar{u}^\perp$, we would have $\bar{u}^\perp =< A_1, \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. On the other hand, since $A_1 \subseteq \bar{v}_4^\perp$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >\subseteq \bar{v}_4^\perp$, we also know that $\bar{v}_4^\perp =< A_1, \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. This implies that $\bar{u} \in< \bar{v}_4 >$, clearly a contradiction. Hence, no nonzero vector of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ is orthogonal with $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. In a similar way, one proves that no nonzero vector of $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ is orthogonal with $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$.

Put $\bar{e}_1 := \bar{v}_1$ and let $\bar{w}_6$ be a vector of $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ for which $f(\bar{e}_1, \bar{w}_6) = 1$. Put $B_2 :=< \bar{v}_4, \bar{w}_6 >$. Then B_1 and B_2 are two distinct 2-spaces of $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ through $< \bar{v}_4 >$. Let $\bar{f}_3$ be a vector of B_1 for which $f(\bar{w}_6, \bar{f}_3) = 1$. Put $A_2 := \bar{f}_3^\perp\cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$. If $A_1 = A_2$, then any nonzero vector of $A_1 = A_2$ orthogonal with $\bar{w}_6$ would be orthogonal with $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >$, contradicting the previous paragraph. Hence, A_1 and A_2 are two distinct 2-spaces of $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >$ through $< \bar{v}_1 >$. Let $\bar{e}_3'$ denote a vector of A_1 for which $f(\bar{e}_3', \bar{f}_3) = 1$ and put $\bar{e}_3 := \bar{e}_3' - f(\bar{e}_3', \bar{w}_6)\cdot\bar{e}_1$. Then $\bar{e}_3 \in A_1$, $f(\bar{e}_3, \bar{f}_3) = 1$ and $f(\bar{e}_3, \bar{w}_6) = 0$. Now, let $\bar{w}_3'$ be a vector of A_2 for which $f(\bar{e}_3, \bar{w}_3') = 1$ and put $\bar{w}_3 := \bar{w}_3' - (f(\bar{w}_3', \bar{w}_6)+1)\cdot\bar{e}_1$. Then $f(\bar{e}_3, \bar{w}_3) = 1$ and $f(\bar{w}_3, \bar{w}_6) = -1$. Now, put $\bar{f}_1 := \bar{w}_6 - \bar{e}_3$ and $\bar{f}_2 := \bar{w}_3 - \bar{f}_3$. Since $\bar{w}_3 \notin \bar{v}_4^\perp$ and $\bar{f}_3 \in \bar{v}_4^\perp$, we have $\bar{f}_2 \notin \bar{v}_4^\perp$. Let $\bar{e}_2 \in< \bar{v}_4 >$ such that $f(\bar{e}_2, \bar{f}_2) = 1$. We can now easily verify that $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f). Moreover, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \lambda_1 \cdot \bar{e}_1 \wedge \bar{e}_3 \wedge (\bar{f}_2 + \bar{f}_3) + \lambda_2 \cdot \bar{e}_2 \wedge \bar{f}_3 \wedge (\bar{f}_1 + \bar{e}_3)$ for some $\lambda_1, \lambda_2 \in \mathbb{F} \setminus \{0\}$. Since $(\lambda_1 \cdot \bar{e}_1, \frac{\bar{f}_1}{\lambda_1}, \frac{\bar{e}_2}{\lambda_1}, \lambda_1 \cdot \bar{f}_2, \frac{\bar{e}_3}{\lambda_1}, \lambda_1 \cdot \bar{f}_3)$ is a hyperbolic basis of (V, f) and $\lambda_1 \cdot \bar{e}_1 \wedge \bar{e}_3 \wedge (\bar{f}_2 + \bar{f}_3) + \lambda_2 \cdot \bar{e}_2 \wedge \bar{f}_3 \wedge (\bar{f}_1 + \bar{e}_3) = (\lambda_1 \cdot \bar{e}_1) \wedge \frac{\bar{e}_3}{\lambda_1} \wedge (\lambda_1 \cdot \bar{f}_3 + \lambda_1 \cdot \bar{f}_2) + (\lambda_1\lambda_2) \cdot \frac{\bar{e}_2}{\lambda_1} \wedge (\lambda_1 \cdot \bar{f}_3) \wedge (\frac{\bar{f}_1}{\lambda_1} + \frac{\bar{e}_3}{\lambda_1})$, we see that the lemma is valid. ∎

The following is a consequence of Lemmas 5.1, 5.2, 5.3 and 5.4.

Corollary 5.5 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f). Then every trivector of Type (C) of V is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

$(C1)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;

$(C2)$ $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;

$(C3)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;

$(C4)$ $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;

$(C5)$ $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$;

$(C6)$ *(only if $|\mathbb{F}| > 2$)* $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F} \setminus \{0\}$ *and some* $\epsilon \in \mathbb{F} \setminus \{0, -1\}$.

Definition 5.6 A trivector of V is said to be of *Type (Ci)*, $i \in \{1, 2, \ldots, 6\}$, if it is $Sp(V, f)$-equivalent with the trivector described in (Ci) of Corollary 5.5.

Lemma 5.7 *Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ and $\{\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{v}'_4, \bar{v}'_5, \bar{v}'_6\}$ be two sets of six linearly independent vectors of V. If $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$, then precisely one of the following cases occurs:*

(1) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >=< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$, $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >=< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3$ *and* $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$;

(2) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >=< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$, $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >=< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$ *and* $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3$.

Proof. By De Bruyn [6, Section 7.3], the set of all vectors $\bar{x}$ for which $(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6) \wedge \bar{x}$ is decomposable is equal to $< \bar{v}_1, \bar{v}_2, \bar{v}_3 > \cup < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$. Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$, this set is also equal to $< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 > \cup < \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$. Hence, either ($< \bar{v}_1, \bar{v}_2, \bar{v}_3 >=< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >=< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$) or ($< \bar{v}_1, \bar{v}_2, \bar{v}_3 >=< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >=< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$).

Suppose the former case occurs. Since $< \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 >$ and $< \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 >$ are two distinct 1-spaces of $\bigwedge^3 V$, $\bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 \in < \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 >$, $\bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6 \in < \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 >$ and $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$, we necessarily have $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3$ and $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$. So, case (1) of the lemma occurs.

If $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >=< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >=< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$, then one proves in a similar way that case (2) of the lemma occurs. ■

Lemma 5.8 *Let $i, j \in \{1, 2, \ldots, 6\}$ with $i \neq j$. Then no trivector of Type (Ci) is $Sp(V, f)$-equivalent with a trivector of Type (Cj).*

Proof. Suppose $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ and $\bar{v}'_1 \wedge \bar{v}'_2 \wedge \bar{v}'_3 + \bar{v}'_4 \wedge \bar{v}'_5 \wedge \bar{v}'_6$ are two $Sp(V, f)$-equivalent trivectors, and that $< \bar{v}_2, \bar{v}_3 > \subseteq \bar{v}_1^{\perp}$, $< \bar{v}_5, \bar{v}_6 > \subseteq \bar{v}_4^{\perp}$, $< \bar{v}'_2, \bar{v}'_3 > \subseteq (\bar{v}'_1)^{\perp}$ and $< \bar{v}'_5, \bar{v}'_6 > \subseteq (\bar{v}'_4)^{\perp}$. There exists a $\theta \in Sp(V, f)$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \theta(\bar{v}'_1) \wedge \theta(\bar{v}'_2) \wedge \theta(\bar{v}'_3) + \theta(\bar{v}'_4) \wedge \theta(\bar{v}'_5) \wedge \theta(\bar{v}'_6)$. By Lemma 5.7, precisely one of the following cases occurs:

(I) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >= \theta(< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >)$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >= \theta(< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >)$;

(II) $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >= \theta(< \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >)$ and $< \bar{v}_4, \bar{v}_5, \bar{v}_6 >= \theta(< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >)$.

So, if one/two of the subspaces $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >, < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ are totally isotropic, then also one/two of the subspaces $< \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >, < \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$ are totally isotropic. This implies that the lemma holds if $\{i, j\} \cap \{1, 2\} \neq \emptyset$. So, we may suppose that $3 \leq i, j \leq 6$ and that none of the subspaces $< \bar{v}_1, \bar{v}_2, \bar{v}_3 >, < \bar{v}_4, \bar{v}_5, \bar{v}_6 >, < \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >, < \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$ is totally isotropic. If case (I) occurs, then we necessarily have $< \bar{v}_1 >= \theta(< \bar{v}'_1 >)$ and $< \bar{v}_4 >= \theta(\langle \bar{v}'_4 \rangle)$. If case (II) occurs, then we necessarily have $< \bar{v}_1 >= \theta(< \bar{v}'_4 >)$ and $< \bar{v}_4 >= \theta(< \bar{v}'_1 >)$. In any case, we have $\bar{v}_1 \in \bar{v}_4^{\perp}$ if and only if $\bar{v}'_1 \in (\bar{v}'_4)^{\perp}$. This implies that the lemma holds if $5 \in \{i, j\}$. So, we may suppose that $\{i, j\} \subseteq \{3, 4, 6\}$ and that $\bar{v}_1 \notin \bar{v}_4^{\perp}$, $\bar{v}'_1 \notin (\bar{v}'_4)^{\perp}$. Now, put $A := \bar{v}_4^{\perp} \cap < \bar{v}_1, \bar{v}_2, \bar{v}_3 >$, $A' := (\bar{v}'_4)^{\perp} \cap < \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 >$, $B := \bar{v}_1^{\perp} \cap < \bar{v}_4, \bar{v}_5, \bar{v}_6 >$ and $B' := (\bar{v}'_1)^{\perp} \cap < \bar{v}'_4, \bar{v}'_5, \bar{v}'_6 >$. Regardless of whether case (I) or case (II) occurs, we always have $\{A,B\}=\{\theta(A'),\theta(B')\}$. So, we have $B\subseteq A^\perp \Leftrightarrow A\subseteq B^\perp \Leftrightarrow B'\subseteq (A')^\perp \Leftrightarrow A'\subseteq (B')^\perp$, $\dim(B\cap A^\perp)=1 \Leftrightarrow \dim(A\cap B^\perp)=1 \Leftrightarrow \dim(A'\cap (B')^\perp)=1 \Leftrightarrow \dim(B'\cap (A')^\perp)=1$ and $A^\perp\cap B=0 \Leftrightarrow B^\perp\cap A=0 \Leftrightarrow (A')^\perp\cap B'=0 \Leftrightarrow (B')^\perp\cap A'=0$. This implies that the lemma also holds if $\{i,j\}\subseteq\{3,4,6\}$. ∎

Lemma 5.9 *Let $(\bar e_1^*,\bar f_1^*,\bar e_2^*,\bar f_2^*,\bar e_3^*,\bar f_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda,\lambda'\in\mathbb{F}\setminus\{0\}$. Then the trivectors $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ and $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda'\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ are $Sp(V,f)$-equivalent if and only if $\lambda'\in\{\lambda,-\lambda\}$.*

Proof. Since $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=(\lambda\bar f_1^*)\wedge\bar f_2^*\wedge\bar f_3^*+(-\lambda)\cdot(-\frac{\bar e_1^*}{\lambda})\wedge(-\bar e_2^*)\wedge(-\bar e_3^*)$ and $(\lambda\bar f_1^*,-\frac{\bar e_1^*}{\lambda},\bar f_2^*,-\bar e_2^*,\bar f_3^*,-\bar e_3^*)$ is a hyperbolic basis of (V,f), we see that the trivectors $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ and $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+(-\lambda)\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ are $Sp(V,f)$-equivalent.

Conversely, suppose that $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ and $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda'\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*$ are $Sp(V,f)$-equivalent. Then there exists a hyperbolic basis $(\bar e_1,\bar f_1,\bar e_2,\bar f_2,\bar e_3,\bar f_3)$ of (V,f) such that $\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*+\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=\bar e_1\wedge\bar e_2\wedge\bar e_3+\lambda'\cdot\bar f_1\wedge\bar f_2\wedge\bar f_3$. By Lemma 5.7, either ($\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*=\bar e_1\wedge\bar e_2\wedge\bar e_3$ and $\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=\lambda'\cdot\bar f_1\wedge\bar f_2\wedge\bar f_3$) or ($\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*=\lambda'\cdot\bar f_1\wedge\bar f_2\wedge\bar f_3$ and $\lambda\cdot\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=\bar e_1\wedge\bar e_2\wedge\bar e_3$). In the former case, we have $\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*\wedge\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=\lambda'\cdot\bar e_1\wedge\bar e_2\wedge\bar e_3\wedge\bar f_1\wedge\bar f_2\wedge\bar f_3$ and hence $\lambda=\lambda'$ by Lemma 2.3. In the latter case, we have $\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar e_3^*\wedge\bar f_1^*\wedge\bar f_2^*\wedge\bar f_3^*=\lambda'\cdot\bar f_1\wedge\bar f_2\wedge\bar f_3\wedge\bar e_1\wedge\bar e_2\wedge\bar e_3$ and hence $\lambda=-\lambda'$ by Lemma 2.3. ∎

Lemma 5.10 *Let $(\bar e_1^*,\bar f_1^*,\bar e_2^*,\bar f_2^*,\bar e_3^*,\bar f_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda,\lambda'\in\mathbb{F}\setminus\{0\}$. Then the trivectors $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)+\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*$ and $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)+\lambda'\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*$ are $Sp(V,f)$-equivalent if and only if $\lambda'=\lambda$.*

Proof. Suppose that $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)+\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*$ and $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)+\lambda'\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*$ are $Sp(V,f)$-equivalent. Then there exists a hyperbolic basis $(\bar e_1,\bar f_1,\bar e_2,\bar f_2,\bar e_3,\bar f_3)$ of (V,f) such that $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)+\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*=\bar f_1\wedge(\bar e_2+\bar e_3)\wedge(\bar f_2-\bar f_3)+\lambda'\cdot\bar e_1\wedge\bar e_2\wedge\bar f_2$. By Lemma 5.7, $<\bar f_1^*,\bar e_2^*+\bar e_3^*,\bar f_2^*-\bar f_3^*>$ is equal to either $<\bar f_1,\bar e_2+\bar e_3,\bar f_2-\bar f_3>$ or $<\bar e_1,\bar e_2,\bar f_2>$. Since $<\bar f_1^*,\bar e_2^*+\bar e_3^*,\bar f_2^*-\bar f_3>$ is totally isotropic and $<\bar e_1,\bar e_2,\bar f_2>$ is not, we necessarily have $<\bar f_1^*,\bar e_2^*+\bar e_3^*,\bar f_2^*-\bar f_3^*>=<\bar f_1,\bar e_2+\bar e_3,\bar f_2-\bar f_3>$. So by Lemma 5.7, $\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)=\bar f_1\wedge(\bar e_2+\bar e_3)\wedge(\bar f_2-\bar f_3)$ and $\lambda\cdot\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*=\lambda'\cdot\bar e_1\wedge\bar e_2\wedge\bar f_2$. Hence, $\lambda\cdot\bar f_1^*\wedge(\bar e_2^*+\bar e_3^*)\wedge(\bar f_2^*-\bar f_3^*)\wedge\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*=\lambda'\cdot\bar f_1\wedge(\bar e_2+\bar e_3)\wedge(\bar f_2-\bar f_3)\wedge\bar e_1\wedge\bar e_2\wedge\bar f_2$, i.e. $-\lambda\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*\wedge\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*=-\lambda'\cdot\bar f_1\wedge\bar e_3\wedge\bar f_3\wedge\bar e_1\wedge\bar e_2\wedge\bar f_2$. This implies that $\lambda=\lambda'$ by Lemma 2.3. ∎

Lemma 5.11 *Let $(\bar e_1^*,\bar f_1^*,\bar e_2^*,\bar f_2^*,\bar e_3^*,\bar f_3^*)$ be a given hyperbolic basis of (V,f) and let $\lambda,\lambda'\in\mathbb{F}\setminus\{0\}$. Then the trivectors $\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*+\lambda\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*$ and $\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*+\lambda'\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*$ are $Sp(V,f)$-equivalent if and only if $\lambda'\in\{\lambda,-\lambda\}$.*

Proof. Since $\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*+\lambda\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*=(\lambda\bar f_1^*)\wedge\bar e_3^*\wedge\bar f_3^*+(-\lambda)\cdot(-\frac{\bar e_1^*}{\lambda})\wedge\bar e_2^*\wedge\bar f_2^*$ and $(\lambda\bar f_1^*,-\frac{\bar e_1^*}{\lambda},\bar e_3^*,\bar f_3^*,\bar e_2^*,\bar f_2^*)$ is a hyperbolic basis of (V,f), we see that the trivectors $\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*+\lambda\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*$ and $\bar e_1^*\wedge\bar e_2^*\wedge\bar f_2^*+(-\lambda)\cdot\bar f_1^*\wedge\bar e_3^*\wedge\bar f_3^*$ are $Sp(V,f)$-equivalent.

Conversely, if the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda' \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ are $Sp(V, f)$-equivalent, then a similar argument as the one of the proof of Lemma 5.9 shows that $\lambda' \in \{\lambda, -\lambda\}$. ■

Lemma 5.12 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f) and let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$. Then the trivectors $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

Proof. Since $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* = (\lambda \cdot \bar{e}_1^*) \wedge (-\bar{e}_2^*) \wedge (\bar{e}_3^* + (-\bar{e}_3^* - \bar{f}_2^*)) + (-\lambda) \cdot (-\frac{\bar{f}_1^*}{\lambda}) \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*)$ and $(-\frac{\bar{f}_1^*}{\lambda}, \lambda \cdot \bar{e}_1^*, \bar{e}_3^*, \bar{e}_2^* + \bar{f}_3^*, -\bar{e}_2^*, -\bar{e}_3^* - \bar{f}_2^*)$ is a hyperbolic basis of (V, f), we see that the trivectors $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + (-\lambda) \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent.

Conversely, if the trivectors $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent, then a similar argument as the one of the proof of Lemma 5.9 shows that $\lambda' \in \{\lambda, -\lambda\}$. ■

Lemma 5.13 *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f) and let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$. Then the trivectors $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ and $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda' \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ are $Sp(V, f)$-equivalent if and only if $\lambda' \in \{\lambda, -\lambda\}$.*

Proof. Since $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*) = \bar{e}_1 \wedge \bar{e}_3 \wedge (\bar{f}_3 + \bar{f}_2) + (-\lambda) \cdot \bar{e}_2 \wedge \bar{f}_3 \wedge (\bar{f}_1 + \bar{e}_3)$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3) := (-\lambda \cdot \bar{e}_2^*, -\frac{\bar{f}_2^*}{\lambda}, \frac{\bar{e}_1^*}{\lambda}, \lambda \cdot \bar{f}_1^*, -\frac{\bar{f}_3^*}{\lambda}, \lambda \cdot \bar{e}_3^*)$ is a hyperbolic basis, the trivectors $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ and $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + (-\lambda) \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ are $Sp(V, f)$-equivalent.

Conversely, if the trivectors $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ and $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda' \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ are $Sp(V, f)$-equivalent, then a similar argument as in the proof of Lemma 5.9 shows that $\lambda' \in \{\lambda, -\lambda\}$. ■

Lemma 5.14 *Let $B^* = (\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V, f), let $\epsilon, \epsilon' \in \mathbb{F} \setminus \{0, -1\}$ and let $\lambda, \lambda' \in \mathbb{F} \setminus \{0\}$. Then the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon' \cdot \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent if and only if $\epsilon' = \epsilon$ and $\lambda' \in \{\lambda, -\lambda\}$.*

Proof. Since $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* = \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \epsilon \cdot \bar{f}_3) + (-\lambda) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3) := (\frac{(1+\epsilon) \cdot \bar{f}_1^*}{-\lambda}, \frac{\lambda}{1+\epsilon} \cdot \bar{e}_1^*, \bar{e}_2^* + \bar{e}_3^*, \frac{\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*}{1+\epsilon}, \epsilon \cdot \bar{e}_2^* - \bar{e}_3^*, \frac{\bar{f}_2^* - \bar{f}_3^*}{1+\epsilon})$ is a hyperbolic basis of (V, f), the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + (-\lambda) \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent.

Conversely, suppose that the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon' \cdot \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* = \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \epsilon' \cdot \bar{f}_3) + \lambda' \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$. By Lemma 5.7, we can distinguish two possibilities.

(a) Suppose $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) = \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \epsilon' \cdot \bar{f}_3)$ and $\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* = \lambda' \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$. Then $\lambda \cdot \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) \wedge \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* = \lambda' \cdot \bar{f}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \epsilon' \cdot \bar{f}_3) \wedge \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ and hence $\lambda\epsilon = \lambda\epsilon'$ by Lemma 2.3. We also have $(1+\epsilon)\cdot\bar{f}_1^* = \pi_{B^*}(\bar{f}_1^* \wedge (\bar{e}_2^*+\bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon\cdot\bar{f}_3^*)) = \pi_B(\bar{f}_1 \wedge (\bar{e}_2+\bar{e}_3) \wedge (\bar{f}_2+\epsilon'\cdot\bar{f}_3)) = (1+\epsilon')\cdot\bar{f}_1$ and $\lambda\cdot\bar{e}_1^* = \pi_{B^*}(\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*) = \pi_B(\lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2) = \lambda'\cdot\bar{e}_1$. Hence, $\lambda(1+\epsilon) = f(\lambda\cdot\bar{e}_1^*, (1+\epsilon)\cdot\bar{f}_1^*) = f(\lambda'\cdot\bar{e}_1, (1+\epsilon')\cdot\bar{f}_1) = \lambda'(1+\epsilon')$. Together with $\lambda\epsilon = \lambda'\epsilon'$, this implies that $(\lambda', \epsilon') = (\lambda, \epsilon)$.

(b) Suppose $\bar{f}_1^* \wedge (\bar{e}_2^*+\bar{e}_3^*) \wedge (\bar{f}_2^*+\epsilon\cdot\bar{f}_3^*) = \lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2$ and $\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^* = \bar{f}_1\wedge(\bar{e}_2+\bar{e}_3)\wedge(\bar{f}_2+\epsilon'\cdot\bar{f}_3)$. Then $\lambda\cdot\bar{f}_1^*\wedge(\bar{e}_2^*+\bar{e}_3^*)\wedge(\bar{f}_2^*+\epsilon\cdot\bar{f}_3^*)\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^* = \lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2\wedge\bar{f}_1\wedge(\bar{e}_2+\bar{e}_3)\wedge(\bar{f}_2+\epsilon'\cdot\bar{f}_3)$ and hence $\lambda\epsilon = -\lambda'\epsilon'$ by Lemma 2.3. We also have $(1+\epsilon)\cdot\bar{f}_1^* = \pi_{B^*}(\bar{f}_1^*\wedge(\bar{e}_2^*+\bar{e}_3^*)\wedge(\bar{f}_2^*+\epsilon\cdot\bar{f}_3^*)) = \pi_B(\lambda'\cdot\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2) = \lambda'\cdot\bar{e}_1$ and $\lambda\cdot\bar{e}_1^* = \pi_{B^*}(\lambda\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*) = \pi_B(\bar{f}_1\wedge(\bar{e}_2+\bar{e}_3)\wedge(\bar{f}_2+\epsilon'\cdot\bar{f}_3)) = (1+\epsilon')\cdot\bar{f}_1$. Hence, $-(1+\epsilon)\lambda = f((1+\epsilon)\cdot\bar{f}_1^*, \lambda\cdot\bar{e}_1^*) = f(\lambda'\cdot\bar{e}_1, (1+\epsilon')\cdot\bar{f}_1) = \lambda'(1+\epsilon')$. Together with $\lambda\epsilon = -\lambda'\epsilon'$, this implies that $\epsilon' = \epsilon$ and $\lambda' = -\lambda$. ■

Theorem 1.6 is a consequence of Lemmas 5.8, 5.9, 5.10, 5.11, 5.12, 5.13 and 5.14.

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